

Limitations of Efficient Reducibility to the Kolmogorov Random Strings

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Abstract. We show the following results for polynomial-time reducibility to R_C , the set of Kolmogorov random strings.

1. If $P \neq NP$, then SAT does not dtt-reduce to R_C .
2. If PH does not collapse, then SAT does not n^α -tt-reduce to R_C for any $\alpha < 1$.
3. If PH does not collapse, then SAT does not n^α -T-reduce to R_C for any $\alpha < \frac{1}{2}$.
4. There is a problem in E that does not dtt-reduce to R_C .
5. There is a problem in E that does not n^α -tt-reduce to R_C , for any $\alpha < 1$.
6. There is a problem in E that does not n^α -T-reduce to R_C , for any $\alpha < \frac{1}{2}$.

These results hold for both the plain and prefix-free variants of Kolmogorov complexity and are also independent of the choice of the universal machine.

Keywords.

1. Introduction

Because the Kolmogorov complexity function $C(x)$ is noncomputable, the set

$$R_C = \{x \mid C(x) > |x|\}$$

of Kolmogorov random strings is undecidable. In fact, R_C has no infinite computably enumerable subset. From this and the fact that the complement $\overline{R_C}$ is computably enumerable, Arslanov's completeness criterion implies that R_C is hard for the c.e. sets under Turing reductions. Kummer [7] showed a stronger result: $\overline{H} \leq_{\text{dtt}} R_C$, where \overline{H} is the complement of the halting problem and \leq_{dtt} denotes a disjunctive truth-table reduction. Neither of these reductions from the halting problem to R_C is efficient. This raises the question [1]: what can be efficiently reduced to R_C ?

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Recall that the Kolmogorov complexity [9] of a binary string x is the length of a shortest program that prints x on a universal Turing machine U :

$$C_U(x) = \min\{|p| \mid U(p) \text{ prints } x\}.$$

For the most part, the theory of Kolmogorov complexity does not depend on the choice of the universal machine U : for any two universal machines U and V , C_U and C_V are within an additive constant of each other. As usual, we fix a universal machine U and omit it from the notation, writing $C(x)$ instead of $C_U(x)$. There are, however, situations when the choice of universal machine matters and then we will be explicit with the subscript. We use the notation $P_\tau(A)$ to denote the class of problems that reduce to A by \leq_τ^P -reductions.

Kummer's result [7] implies there is a computable time bound $t(n)$ such that for every decidable A , $A \leq_{\text{dtt}}^{t(n)} R_C$. Kummer's proof is nonconstructive and does not yield any information about the function $t(n)$. In fact, Allender et al. [1] show that some uncertainty about the time bound $t(n)$ is inevitable. They show that the $t(n)$ in Kummer's theorem may be arbitrarily large, depending on the choice of the universal machine U . Formally, for every computable time bound $t(n)$, there exists a universal machine U and a decidable set A such that A does not $\leq_{\text{dtt}}^{t(n)}$ -reduce to R_{C_U} . On the other hand, independent of U , there exist decidable sets with arbitrarily high time complexity that reduce to R_{C_U} via a polynomial-time dtt-reduction: for every computable time bound $t(n)$ and every universal machine U , there is a set $A \in \text{DEC} - \text{DTIME}(t(n))$ such that $A \leq_{\text{dtt}}^P R_{C_U}$. While this result shows $P_{\text{dtt}}(R_C)$ contains sets of high time complexity, the set A in this theorem is constructed via padding, which makes A very sparse. Thus while A has high time complexity, A is very simple in other terms. We show that this simplicity is inherent: any such A is highly predictable in the sense of polynomial-time dimension. From this it follows that R_C is not hard for E under \leq_{dtt}^P -reductions. This holds for every universal machine, i.e. $E \not\subseteq P_{\text{dtt}}(R_{C_U})$ for every U . We also show that R_C is not polynomial-time dtt-hard for NP unless $P = \text{NP}$. Both of these results follow from showing that if a decidable set \leq_{dtt}^P -reduces to R_C , then the set \leq_{dtt}^P -reduces to a tally set. These results complement the result of Allender et al. [1] that

$$P = \text{DEC} \cap \bigcap_U P_{\text{dtt}}(R_{C_U}),$$

where the intersection is over all universal machines. While the class $\text{DEC} \cap P_{\text{dtt}}(R_{C_U})$ contains arbitrarily complex sets, it is intuitively "close" to P for every U , in that it has small dimension and cannot contain NP unless $P = \text{NP}$.

Allender et al. [2] showed that R_C is hard for PSPACE under polynomial-time Turing reductions: $\text{PSPACE} \subseteq P_T(R_C)$. Buhrman et al. [3] showed that R_C is hard for BPP under polynomial-time truth-table reductions: $\text{BPP} \subseteq P_{\text{tt}}(R_C)$. We consider bounded query Turing and truth-table reductions. Based on the Winnow algorithm [10] and polynomial-time dimension [6], we show that R_C is not $\leq_{n^\alpha\text{-tt}}^P$ -hard for E, for any $\alpha < 1$. This is an improvement of a result in [1] which obtained the same consequence for EE. Also, we use the techniques of [5,4]

to show that R_C is not $\leq_{n^{\alpha}\text{-tt}}^{\text{P}}$ -hard for NP unless $\text{NP} \subseteq \text{coNP/poly}$ and the polynomial-time hierarchy collapses by Yap's theorem [13]. Finally, we obtain the same consequences for $\leq_{n^{\alpha}\text{-T}}^{\text{P}}$ -reductions, for all $\alpha < \frac{1}{2}$.

2. Preliminaries

We use standard notions of polynomial-time reducibilities [8]. We also need the following two notions of reducibility.

Definition. Let $\mathcal{B} = (B_n \mid n \geq 0)$ be a family of subsets of Σ^* . We say that A NP-reduces to \mathcal{B} if there is an NPMV function N such that for all n , for all $x \in \Sigma^n$, $x \in A$ iff at least one output of $N(x)$ is in B_n .

Definition. Let $\mathcal{B} = (B_n \mid n \geq 0)$ be a family of subsets of Σ^* . We say that A disjunctively reduces to \mathcal{B} in $t(n)$ time if there is an algorithm M such that for all n , for all $x \in \Sigma^n$, $M(x)$ outputs a list of strings in $t(n)$ time and $x \in A$ iff at least one output of $M(x)$ is in B_n .

The following lemma is from [4], based on a technique of [5]. An AND-function (of order 1) for a set A is a polynomial-time computable function g such that for all strings x_1, x_2, \dots, x_n , $|g(x_1, \dots, x_n)| = O(\sum_{i=1}^n |x_i|)$ and $g(x_1, x_2, \dots, x_n) \in A$ iff $x_i \in A$ for all i .

Lemma 2.1. *Let A have an AND-function and let $\alpha < 1$. Let $\mathcal{B} = (B_n \mid n \geq 0)$ be a family of sets with $|B_n| \leq 2^{n^\alpha}$ for sufficiently large n . If A NP-reduces to \mathcal{B} , then $A \in \text{NP/poly}$.*

The p-dimension [11] of a complexity class is a real number in $[0, 1]$. The p-dimension of P is 0 and the p-dimension of E is 1. For this paper, we do not need the full details of p-dimension; all we require is the fact that a p-dimension 0 class cannot contain E and the following lemma. The proof of this lemma relies on the Winnow online learning algorithm [10] and is straightforward to prove using the approach of [6].

Lemma 2.2. *Let $\alpha < 1$ and let $c \geq 1$. Let X be the class of all A for which there exists a family $\mathcal{B} = (B_n \mid n \geq 0)$ with $|B_n| \leq 2^{n^\alpha}$ such that A disjunctively reduces to \mathcal{B} in 2^{cn} time. Then X has p-dimension 0. In particular, X does not contain E.*

3. Disjunctive Reductions

Theorem 3.1. *If A is decidable and $A \leq_{\text{dtt}}^{\text{P}} R_C$, then $A \leq_{\text{dtt}}^{\text{P}} B$ for some $B \in \text{TALLY}$.*

Proof. We use the proof technique from [1] that A is decidable and $A \leq_{\text{mтт}}^{\text{P}} R_C$ (monotone truth-table) implies $A \in \text{P/poly}$, observing that we can encode in a tally set to obtain the stronger result.

Suppose A is decidable and $A \leq_{\text{dtt}}^{\text{P}} R_C$ via a reduction computable in time n^d . Let the queries on input x be denoted by $Q(x)$. For some constant c , we claim only the queries of length at most $l(n) = c \log n$ “matter.”

For any x , we have $x \in A$ iff $Q(x) \cap R_C \neq \emptyset$. Define $Q'(x) = Q(x) \cap \Sigma^{\leq l(n)}$, where $n = |x|$. We claim that for each $x \in A$, there is some $q \in Q'(x)$ such that for all y with $|y| = |x|$, $q \in Q'(y)$ implies $y \in A$.

Suppose the claim is false. Then given n , we can find the first string x of length n such that $x \in A$ and each query $q \in Q'(x)$ belongs to $Q'(y)$ for some $y \notin A$. This implies that $Q'(x) \cap R_C = \emptyset$. Since $x \in A$, it follows that $Q(x) - Q'(x)$ contains a string $r \in R_C$. This string r has $C(r) > l(n)$ because $r \notin Q'(x)$. We can describe r by describing n and the index of r in $Q(x)$. Since $|Q(x)| \leq n^d$, this takes at most $(d+3) \log n$ bits, a contradiction if we choose $c = d+4$.

Let $\{w_1, \dots, w_N\}$ be an enumeration of $\Sigma^{\leq l(n)}$. Let I_n be the collection of all i where for all y of length n , $w_i \in Q(y)$ implies $y \in A$. Our desired tally set is $\{0^{(n,i)} \mid n \geq 0 \text{ and } i \in I_n\}$, where $\langle \cdot, \cdot \rangle$ is a pairing function on the natural numbers. \square

Corollary 3.2. *If $\text{P} \neq \text{NP}$, then $\text{NP} \not\subseteq \text{P}_{\text{dtt}}(R_C)$.*

Proof. Suppose that $\text{NP} \subseteq \text{P}_{\text{dtt}}(R_C)$. By Theorem 3.1, $\text{SAT} \leq_{\text{dtt}}^{\text{P}} B$ for a tally set B . Then $\overline{\text{SAT}} \leq_{\text{ctt}}^{\text{P}} \overline{B} \cap 0^*$. Ukkonen [12] showed that $\text{P} = \text{NP}$ if coNP has a sparse $\leq_{\text{ctt}}^{\text{P}}$ -hard set. \square

Corollary 3.3. *The class $\text{P}_{\text{dtt}}(R_C) \cap \text{DEC}$ has p-dimension 0.*

Proof. Theorem 3.1 implies $\text{P}_{\text{dtt}}(R_C) \cap \text{DEC} \subseteq \text{P}_{\text{dtt}}(\text{TALLY}) \subseteq \text{P}_{\text{dtt}}(\text{SPARSE})$. This last class was shown to have p-dimension 0 in [6]. \square

Corollary 3.4. $\text{E} \not\subseteq \text{P}_{\text{dtt}}(R_C)$.

Proof. This follows from Corollary 3.3 because E has p-dimension 1. \square

4. Truth-Table Reductions

Theorem 4.1. *Let $\alpha < 1$.*

1. *If A is decidable, A has an AND-function, and $A \leq_{n^\alpha\text{-tt}}^{\text{P}} R_C$, then $A \in \text{NP/poly}$.*
2. *The class $\text{P}_{n^\alpha\text{-tt}}(R_C) \cap \text{DEC}$ has p-dimension 0.*

Proof. The main idea of the proof is from [1]. We expound the argument here and show how to apply Lemmas 2.1 and 2.2.

Let A be decidable such that $A \leq_{n^\alpha\text{-tt}}^{\text{P}} R_C$. Write $Q(x)$ for the truth-table reduction’s queries on input x and $Z_x \subseteq \Sigma^{n^\alpha}$ for the query answer sequences that cause the reduction to accept x . That is, if $Q(x) = \{q_1, \dots, q_{n^\alpha}\}$ in lexicographic order, then $x \in A$ if and only if $R_C[q_1] \cdots R_C[q_{n^\alpha}] \in Z_x$.

Let $l(n) = n^\epsilon$, where $0 < \epsilon < 1 - \alpha$. We claim that the truth-table reduction is still correct if we only use the queries of length at most $l(n)$. Formally, let

$Q'(x) = Q(x) \cap \Sigma^{\leq l(n)}$ and let Z'_x be the restriction of Z_x with bits corresponding to strings in $Q(x) - Q'(x)$ removed.

Call two strings x and y of the same length *equivalent* if $Q'(x) = Q'(y)$. We claim that for each $x \in A$, there is some $z_x \in Z'_x$ such that for all y equivalent to x , $z_x \in Z'_y$ iff $y \in A$.

Suppose the claim is false. We can find the least $x \in A$ such that for all $z \in Z'_x$, there is some y_z equivalent to x such that $z \in Z'_y$ iff $y_z \notin A$. Let v be the correct answer sequence for $Q'(x) \cap R_C$ and let r be the number of 1's in v ; that is, $r = |Q'(x) \cap R_C|$. Given x and r , we can enumerate $\overline{R_C}$ to compute $Q'(x) \cap R_C$ and obtain v . Then we can compute y_v such that query answers v are incorrect for y_v . This means that $Q(y_v) - Q'(y_v)$ must contain a string in R_C with length $> l(n)$. However, we can describe this string by describing n , r , and its index in $Q(y_v)$, which takes $O(\log n)$ bits, a contradiction.

We define a family of sets $\mathcal{B} = (B_n \mid n \geq 0)$ as follows. For each equivalence class $[x]$ with queries $Q'(x) = \{w_1, \dots, w_{n^\alpha}\}$ and $z_x \in Z'_x$ the answer sequence that is correct for all strings in the equivalence class, we put the tuple $\langle w_1, \dots, w_{n^\alpha}, z_x \rangle$ in B_n . Note that $|B_n| < 2^{n^\gamma}$ where $\alpha + \epsilon < \gamma < 1$. By the claim, A NP-reduces to \mathcal{B} . It follows from Lemma 2.1 that $A \in \text{NP/poly}$ if A has an AND-function.

We also have that A is disjointly reducible in 2^n time to \mathcal{B} . Therefore Lemma 2.2 applies to show $P_{n^\alpha\text{-tt}}(R_C) \cap \text{DEC}$ has p-dimension 0. \square

Corollary 4.2. *If $\text{NP} \subseteq P_{n^\alpha\text{-tt}}(R_C)$ for some $\alpha < 1$, then $\text{NP} \subseteq \text{coNP/poly}$.*

Proof. This follows from Theorem 4.1 because the hypothesis implies $\overline{\text{SAT}} \leq_{n^\alpha\text{-tt}}^P R_C$ and $\overline{\text{SAT}}$ has an AND-function. \square

Corollary 4.3. *If the polynomial-time hierarchy does not collapse, then $\text{NP} \not\subseteq P_{n^\alpha\text{-tt}}(R_C)$ for any $\alpha < 1$.*

Proof. This is immediate from Corollary 4.2 and Yap's theorem [13] that $\text{NP} \subseteq \text{coNP/poly}$ implies the polynomial-time hierarchy collapses to its third level. \square

Corollary 4.4. *For any $\alpha < 1$, $\text{E} \not\subseteq P_{n^\alpha\text{-tt}}(R_C)$.*

Proof. This follows from Theorem 4.1 because E has p-dimension 1. \square

5. Turing Reductions

Theorem 5.1. *Let $\alpha < \frac{1}{2}$.*

1. *If A is decidable, A has an AND-function, and $A \leq_{n^\alpha\text{-T}}^P R_C$, then $A \in \text{NP/poly}$.*
2. *The class $P_{n^\alpha\text{-T}}(R_C) \cap \text{DEC}$ has p-dimension 0.*

Proof. Let $\alpha < \beta < \frac{1}{2}$. Suppose that $A \in \text{DEC}$ and $A \leq_{n^\alpha\text{-T}}^P R_C$ via M . Let M' be the Turing machine that simulates M and whenever M makes a query of length at least n^β , M' makes no query and proceeds as if the answer to the query were no. We use the following concepts:

- An *advice* is a tuple $(z, w_1, \dots, w_{n^\alpha})$ such that $z \in \Sigma^{n^\alpha}$ and each $w_i \in \Sigma^{<n^\beta}$.
- A string y is *accepted with advice* $(z, w_1, \dots, w_{n^\alpha})$ if $M'(y)$ queries w_1, \dots, w_{n^α} and accepts y when M' is given $z[1], \dots, z[n^\alpha]$ as the query answers.
- An advice $(z, w_1, \dots, w_{n^\alpha})$ is *safe* if for all $y \in \Sigma^n$, y is accepted with advice $(z, w_1, \dots, w_{n^\alpha})$ implies $y \in A$.

We claim that for all $x \in A_{=n}$, there is a safe advice (z, \vec{w}) such that x is accepted with advice (z, \vec{w}) .

Suppose the claim is false. Then we can find the least $x \in A_{=n}$ that does not have a safe advice. We can specify the correct answer sequence $z \in \Sigma^{n^\alpha}$ for $M(x)$ when querying oracle R_C . With this correct answer sequence z , M must query some string in R_C that is not in $\Sigma^{<n^\beta}$. Therefore we can describe a string r with $C(r) \geq n^\beta$ by describing n , z , and the index of r in $M(x)$'s query set on query answer sequence z . Thus $C(r) \leq n^\alpha + O(\log n)$, which is a contradiction since $\alpha < \beta$.

We define a family of sets \mathcal{B} by putting into B_n all advices $(z, w_1, \dots, w_{n^\alpha})$ that are safe. Let $1 > \gamma > \alpha + \beta$. The total number of possible advices is at most $2^{n^\alpha} \cdot (2^{n^\beta})^{n^\alpha} < 2^{n^\gamma}$, so $|B_n| < 2^{n^\gamma}$. We have that A NP-reduces to \mathcal{B} and A disjunctively reduces in 2^n time to \mathcal{B} , so the theorem follows from Lemmas 2.1 and 2.2. \square

Corollary 5.2. *If $\text{NP} \subseteq \text{P}_{n^\alpha\text{-T}}(R_C)$ for some $\alpha < \frac{1}{2}$, then $\text{NP} \subseteq \text{coNP}/\text{poly}$.*

Corollary 5.3. *If the polynomial-time hierarchy does not collapse, then $\text{NP} \not\subseteq \text{P}_{n^\alpha\text{-T}}(R_C)$ for any $\alpha < \frac{1}{2}$.*

Corollary 5.4. *For any $\alpha < \frac{1}{2}$, $\text{E} \not\subseteq \text{P}_{n^\alpha\text{-T}}(R_C)$.*

6. Open Problems

We believe the following open problems should be tractable but appear to require techniques beyond those used in this paper.

Problem 6.1. *Show that $\text{E} \not\subseteq \text{P}_{n^\alpha\text{-T}}(R_C)$ for $\frac{1}{2} \leq \alpha < 1$.*

Problem 6.2. *Show that $\text{NP} \not\subseteq \text{P}_{n^\alpha\text{-T}}(R_C)$ for $\frac{1}{2} \leq \alpha < 1$ under a reasonable hypothesis.*

It is unknown whether even every decidable problem is polynomial-time Turing reducible to R_C . We conjecture that $\text{ESPACE} \not\subseteq \text{P}_{\text{T}}(R_C)$ and that this can be proved using resource-bounded dimension or measure:

Problem 6.3. *Show that $\text{P}_{\text{T}}(R_C) \cap \text{DEC}$ has pspace-dimension 0.*

Lastly, we know $\text{SAT} \leq_{\text{dtt}} R_C$ and that $\text{SAT} \leq_{\text{dtt}}^{\text{P}} R_C$ iff $\text{P} = \text{NP}$. What more can be said about the amount of time it takes to disjunctively reduce SAT to R_C ?

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